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MODEL OF VORTEX RING FORMATION

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A semi-empirical model of vortex ring formation during exhaustion of a pulsed submerged jet from a circular orifice is presented. Formulas for determining the parameters of the vortex ring, depending on the conditions of formation of the latter, are derived. The calculated characteristics of the vortex ring as functions of criteria determining the vortex formation process are demonstrated to be in good agreement with experimental data.

Key words: *vortex ring, exhaustion of a submerged jet.*

Introduction. Vortex rings as one form of vortex motion of liquids and gases have been studied for more than one hundred years. The vortex ring is a toroidal volume of a swirled fluid, which moves in the ambient medium perpendicular to the ring plane [1]. The fluid motion is axisymmetric, and the vorticity vector (velocity rotor) in the torus is directed along the circumferences coaxial with the circumferential axis of the torus. The cross-sectional shape of thin vortex rings is close to a circle. The toroidal vortex ring captures a certain volume of the fluid around the ring, which moves together with the vortex ring and has the shape similar to an ellipsoid of revolution flattened in the direction of ring motion (Fig. 1). This closed volume of the fluid is called the vortex atmosphere. For a long time, the ideas about the structure and the laws of motion of vortex rings were based on mathematical models of a vortex ring, which were developed for an ideal fluid [1–3]. Experimental studies allowed obtaining more reliable information on the structure and processes of formation of real vortex rings [4–7].

Knowing the laws of vortex ring formation is necessary to produce vortex rings with prescribed characteristics and to develop methods for calculating the parameters of the vortex ring, depending on conditions of formation of the latter. Vortex rings can be generated by different methods. One method implies that a flat circular disk submerged into a fluid is suddenly pushed and set into motion in the direction perpendicular to the disk plane; after that, the disk is immediately removed from the fluid. Vortex rings of this kind (more exactly, half-rings) can be observed, for instance, on the surface of coffee poured into a cup, which is rapidly stirred by a small spoon half-submerged into the cup. Through simple considerations, Taylor [8] derived formulas for estimating the parameters of the vortex ring formed owing to disk motion, depending on the disk radius and its velocity. Such a method of vortex ring generation, however, can be considered as a speculative experiment or as a method for production of vortex rings for qualitative research only, because it is next to impossible in real experiments to ensure controlled motion and subsequent removal of the disk from the fluid without disturbing the resultant flow. Hence, vortex rings are mainly generated by another method, with the use of exhaustion of a finite-length submerged jet from a circular orifice or from an open end of a cylindrical tube.

Theoretical calculations of the parameters of vortex rings formed during exhaustion of a submerged jet is a more complicated problem. This problem attracted the attention of many researchers [9, 10]. Saffman [9] considered the model of vortex ring formation from a finite-length cylindrical vortex sheet in an unbounded fluid. This model, however, is inconsistent with the real pattern of vortex ring formation, because the cylindrical sheet is not formed during jet exhaustion: the vortex sheet immediately starts to roll up into a toroidal spiral, and a vortex ring is formed. Vladimirov and Tarasov [10] proposed a physically more grounded model of vortex ring formation. Based on the laws of conservation of impulse and energy and additional assumptions on the distribution of circulation

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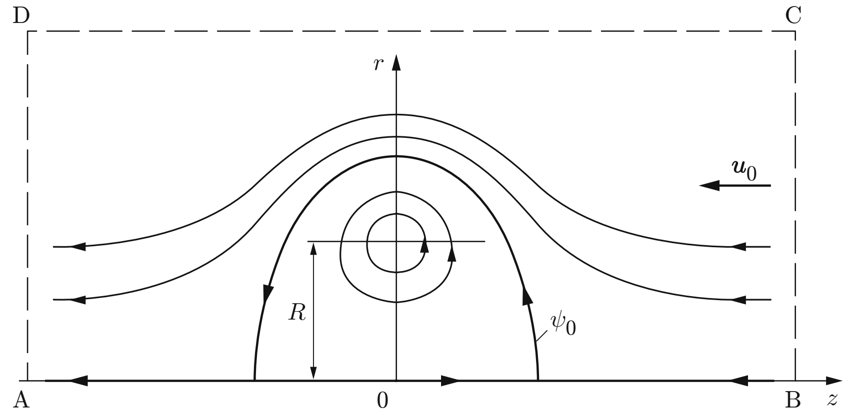

Fig. 1. Streamlines in the vortex ring: ψ_0 is the streamline bounding the vortex atmosphere; ABCD is the contour of integration for determining the vortex ring circulation.

in the vortex core, Vladimirov and Tarasov [10] determined the changes in some parameters of the vortex ring in the course of jet exhaustion. Their model provides a qualitatively correct idea about the dynamics of vortex ring formation and the evolution of some characteristics of the vortex ring during jet exhaustion. In deriving the formulas, however, they used the law of energy conservation with no allowance for dissipation of some part of the jet energy in the course of vortex formation. The fact that the jet impulse is not completely transformed to the vortex ring impulse is also ignored. Therefore, the available formulas do not allow exact quantification of the parameters of the vortex ring formed.

Formulation of the Problem. A model of formation of the vortex ring during exhaustion of a submerged jet is presented below. The model is based on more correct application of conservation laws. Formulas for almost all characteristics of the vortex ring are derived and found to be in good agreement with experimental data. The results of experimental investigations of the structure and parameters of vortex rings formed owing to exhaustion of a submerged jet with a constant velocity V_0 during a finite time T from a cylindrical nozzle of radius R_0 and length l show that the vortex ring formation is mainly determined by two dimensionless criteria [4]:

$$L_* = V_0 T / R_0, \qquad \mathrm{Re} = V_0 R_0 / \nu. \tag{1}$$

Here L_* is the dimensionless jet length (or the duration of jet exhaustion), Re is the Reynolds number of the jet, and ν is the kinematic viscosity of the medium.

The vortex ring structure substantially depends on the distribution of vorticity; according to the data [4], however, the structure of the turbulent vortex ring formed can be characterized in a simplified formulation by a finite number of parameters: ring radius R, cross-sectional radius of the vortex core a, circulation Γ, translational velocity u_0, impulse P, and energy E. The vortex ring parameters were determined in [4] by measuring the velocity field with the help of two hot-wire probes mounted on the way of vortex motion at a certain distance from the nozzle exit, where the vortex formation process can be assumed to be completed. It should be noted that most of these parameters can be determined in experiments without a detailed analysis of the global distributions of the velocity field or vorticity. The translational velocity u_0 of the vortex ring is determined by means of vortex visualization and photographing of its motion as a function of time. The geometric parameters R and a are determined from the distribution of the velocity u in the vortex ring plane ($z = 0$).

Figure 2 shows the distribution of the velocity u in the vortex ring plane in a coordinate system where the vortex moves with a velocity u_0 with respect to the fluid, which is at rest at infinity. The velocity distribution can be plotted in a vortex-fixed coordinate system if the dashed line $u = u_0$ is used as the r axis in Fig. 2. In this system, the distance between the origin and the point of intersection of the curve $u(0, r)$ with the abscissa axis determines the vortex ring radius R. The linear segment of the curve $u(0, r)$ in the neighborhood of the point $r = R$ corresponds to the swirled vortex core, and the distance $2a$ along the r axis between the maximum and minimum points at the ends of the linear segment of the curve can be used as the transverse size (diameter) of the vortex core.

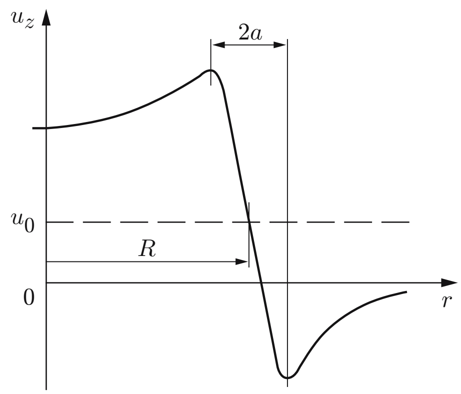

Fig. 2. Distribution of velocity in the vortex ring plane.

The velocity circulation $\Gamma = \oint u\,dl$ in the vortex ring is determined by integration of the velocity distribution over a closed rectangular contour ABCD whose side AB coincides with the axis of symmetry of the vortex ring (see Fig. 1). If the rectangle size is sufficiently large, then the flow velocity on the sides BC, CD, and DA of the rectangle is equal to zero in a coordinate system where the fluid is at rest at infinity, and it suffices to integrate the velocity distribution along the axis of symmetry only (along the line AB) to calculate the circulation. Assuming that the translational velocity of the vortex ring u_0 remains unchanged during the registration of the velocity distribution on the axis of symmetry, we can demonstrate that the circulation is determined from the directly measured distribution of the velocity u on the axis of symmetry as a function of the time t:

$$\Gamma = \oint\limits_{\text{ABCD}} u(z,r)\,dl = \int\limits_{\text{AB}} u(z,0)\,dz = u_0 \int\limits_{-\infty}^{\infty} u(t)\,dt$$

($z = -u_0 t$). As the velocimeter probe is assumed to be placed on the axis of symmetry at a sufficiently large distance from the approaching vortex ring, the lower limit of the integral with respect to t can be taken equal to zero.

The value of the vortex impulse P is determined on the basis of results obtained by Vladimirov [11] who showed that, for the vorticity concentrated in a finite closed volume V, the vortex impulse can be presented as a sum of the impulse of this volume of fluid and its virtual impulse. Hence, by choosing the volume of the vortex ring atmosphere as the volume V, we can present the impulse as

$$\boldsymbol{P} = \rho V(1+\mu)\boldsymbol{u}_0,$$

where ρ is the medium density, $\boldsymbol{u}_0$ is the velocity of the vortex ring, and μ is the coefficient of the virtual mass of the vortex atmosphere in the direction of its motion. The vortex ring energy E can also be presented as a sum of the energy of the vortex ring atmosphere $E_a = \rho \int\limits_V \frac{|u|^2}{2}\,dV$ and the energy of the virtual mass of the vortex atmosphere μE_a. Unfortunately, there are no experimental data on the dependence of the vortex ring energy on criteria (1). The experimental results in [4] were presented in the form of the dependence of the dimensionless parameters of the vortex ring

$$R_* = \frac{R}{R_0}, \quad a_* = \frac{a}{R_0}, \quad u_0^* = \frac{u_0}{V_0}, \quad \Gamma_* = \frac{\Gamma}{R_0 V_0}, \quad P_* = \frac{P}{\pi\rho R_0^3 V_0} \tag{2}$$

on the dimensionless criteria (1) governing the vortex formation process.

The model of vortex ring formation presented in the paper is based on the approaches used in [4], where the problem formulation was given and the principles of estimating some parameters of the vortex ring were described. Some important vortex ring parameters characterizing its structure, however, were not identified. The present paper offers a technique for calculating all parameters (2) as functions of the dimensionless criteria (1) determining the process of vortex ring formation.

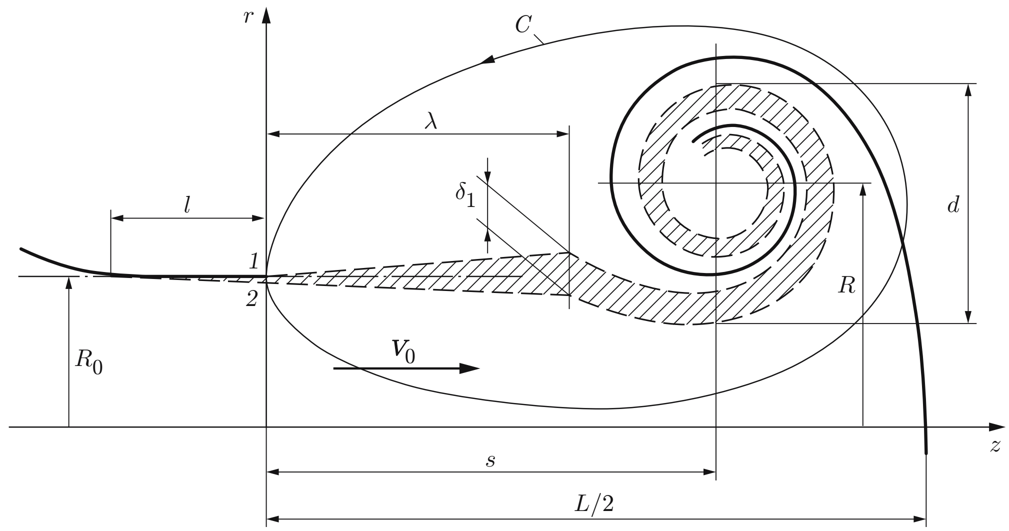

Fig. 3. Pattern of vortex ring formation.

A pattern of vortex ring formation during exhaustion of a submerged jet, which is consistent with experimental observations, is schematically shown in Fig. 3. The forming vortex passed to a distance s from the tube exit and is fed by the jet escaping from the tube. At a distance λ from the tube exit, the jet is almost undisturbed and has the form of a cylinder. Vortex ring formation is considered in a cylindrical coordinate system (z, r). According to Fig. 3, using the laws of conservation of impulse and circulation and involving some kinematic considerations, we can derive formulas for determining almost all parameters characterizing the structure of the vortex ring formed as functions of the jet length and Reynolds number. The forming vortex ring was assumed to be thin in calculations, which corresponds to high values of the jet Reynolds number ($\mathrm{Re} = R_0V_0/\nu \geqslant 10^4$). It is also assumed that the relations between the ring parameters are determined by the formulas of theoretical models of the vortex ring in an ideal fluid [1–3].

Vortex Impulse. The impulse of the vortex ring can be determined by estimating the vortex impulse in the fluid during jet exhaustion. The vortex impulse is known to be determined by the integral [1]

$$\boldsymbol{P} = \frac{1}{2}\,\rho \int\limits_{\tau} (\boldsymbol{r} \times \boldsymbol{\omega})\, d\tau,$$

where $\boldsymbol{r}$ is the radius vector, $\boldsymbol{\omega}$ is the vorticity, and τ is the volume of integration corresponding to the ambient space outside the tube. Hence, the rate of impulse changes in a motionless volume τ is determined by the expression

$$\frac{\partial \boldsymbol{P}}{\partial t} = \frac{1}{2}\rho \int\limits_{\tau} \Big(\boldsymbol{r} \times \frac{\partial \boldsymbol{\omega}}{\partial t}\Big)\, d\tau,$$

the right side of this expression being brought with the use of the Helmholtz equation $\partial\boldsymbol{\omega}/\partial t = \mathrm{rot}\,(\boldsymbol{u} \times \boldsymbol{\omega})$ to an integral over the surface Σ bounding the fluid outside the tube:

$$\frac{\partial \boldsymbol{P}}{\partial t} = \frac{1}{2}\,\rho \iint\limits_{\Sigma} \Big\{\boldsymbol{n}[\boldsymbol{r} \cdot (\boldsymbol{u} \times \boldsymbol{\omega})] - (\boldsymbol{n} \cdot \boldsymbol{r})(\boldsymbol{u} \times \boldsymbol{\omega}) + \boldsymbol{n}q^2 - 2(\boldsymbol{n}\boldsymbol{u})\boldsymbol{u}\Big\}\, d\Sigma.$$

Here $q = |\boldsymbol{u}|$ is the absolute value of velocity and $\boldsymbol{n}$ is the outward normal to the surface Σ. The surface Σ consists of the tube exit section, the outer surface of the tube, and the surface located at an infinite distance from the tube exit, which includes the entire space filled by the fluid outside the tube.

In calculations, we can assume that $\boldsymbol{u} = \boldsymbol{\omega} = 0$ on the outer surface of the tube. Then, the integral over the tube surface vanishes. The integral over the surface infinitely distant from the tube exit is also equal to zero, because $\boldsymbol{\omega} = 0$ at infinity, and the velocity $\boldsymbol{u}$ has an asymptotic behavior of a source (i.e., $|\boldsymbol{u}| \sim 1/|\boldsymbol{r}|^2$ as $|\boldsymbol{r}| \to \infty$). There remains only the integral over the tube exit cross section, which has two characteristic zones: the zone of exhaustion of the central core of the jet with $\boldsymbol{u} = \boldsymbol{V}_0$ and $\boldsymbol{\omega} = 0$ and the ring-shaped zone of exhaustion of the boundary layer of thickness δ from the inner wall of the tube. We have $\boldsymbol{u} = \boldsymbol{V}_0$ on the inner cylindrical surface of the boundary layer adjacent to the jet core and $\boldsymbol{u} = 0$ on the outer boundary ($r = R_0$). Taking into account that $\delta/R_0 \ll 1$, we can assume that $\boldsymbol{u} = \boldsymbol{k}V_0/2$ and $\boldsymbol{\omega} = \boldsymbol{e}V_0/\delta$ on the average over the entire cross section of the boundary layer ($\boldsymbol{k}$ is the unit vector along the z axis and $\boldsymbol{e}$ is the unit vector directed along the azimuthal coordinate). During integration, terms of the order δ/R_0 and higher are rejected as being negligibly small. It follows from calculations that the vector $\partial \boldsymbol{P}/\partial t$ has a value $\partial P_z/\partial t = \pi\rho R_0^2 V_0^2$; hence, the impulse in the fluid at the time $t = T$ is

$$P_z = \int\limits_0^T \frac{\partial P_z}{\partial t}\, dt = \pi\rho R_0^2 V_0^2 T.$$

According to Fig. 3, there are two vortex structures in the fluid at the time $t = T$: the forming vortex ring and the rear part of the cylindrical vortex sheet of length λ whose impulse equals $P_\lambda = \pi\rho R_0^2 \lambda V_0$. It follows from experiments that the rear part of the vortex sheet at $t > T$ transforms to a small secondary vortex ring with an impulse P_1 on the rear front of the jet, which soon dissipates. Approximately, we can assume that $P_1 = \alpha P_\lambda$ (α is the proportionality coefficient of the order of unity). Thus, for $t > T$, the impulse of the fluid P_z is a sum of the impulses of the main vortex ring P and the secondary vortex structure P_1 on the rear front of the jet, i.e., $P_z = P + P_1$. Thus, the impulse of the vortex ring is $P = P_z - P_1 = \pi\rho R_0^2 V_0^2 T - \alpha\pi\rho R_0^2 \lambda V_0$. Theoretical models of the vortex ring [1–3] predict that the vortex ring impulse with a thin core is related to the circulation Γ and the ring radius R by the formula $P \approx \pi\rho R^2 \Gamma$. Substituting this formula into the above-derived expression, we can find the relation of R and Γ to the jet parameters V_0, T, and λ:

$$R^2\Gamma = R_0^2 V_0^2 T - \alpha R_0^2 V_0 \lambda. \tag{3}$$

Velocity Circulation. The vortex ring circulation can be determined as a function of the jet parameters by estimating the total velocity circulation Γ_C arising as the jet escapes from the tube into the ambient space. In estimating the velocity circulation Γ_C, one should take into account that the jet velocity near the nozzle exit during a small initial period $t_0 \ll R_0/V_0$ is not equal to V_0, because the intensity γ_0 of the vortex sheet formed on the inner wall of the tube is not constant along the tube generatrix at $t = 0$. The calculations of the potential flow arising at the initial time [12] imply that the distribution of γ_0 near the tube edge (at $z \approx 0$) has the form $\gamma_0 \approx A/\sqrt{z}$, where $A = \text{const}$ and z is the distance from the exit cross section inward the tube. At this stage, the flow near the tube edge is self-similar and is determined only by the coefficient A. The self-similar stage of vortex sheet roll-up was considered in [13–15].

Nevertheless, the velocity circulation Γ_C arising in the fluid at large times of jet exhaustion ($T \gg t_0$) can be estimated without analyzing the initial stage of vortex sheet evolution in detail. The effect of the self-similar stage of vortex sheet roll-up on circulation in the fluid can be estimated by assuming that the velocity circulation Γ_C at the time $t = 0$ already acquires a certain finite value γ, and the further growth of circulation in the fluid proceeds with a constant velocity of jet exhaustion. The quantity γ can be considered as an empirical constant whose value at $T > R_0/V_0$ is small, as compared with Γ_C.

It should be noted that Fig. 3 predicts that the only vortex region in the fluid is the vortex sheet displaying spiral motion. If we make a cut along the vortex sheet and along the tube generatrix, the flow in the entire space bounded by the cut surfaces can be assumed to be irrotational and to have a velocity potential φ. Let us calculate the velocity circulation over the motionless contour C, which covers the spiral end of the vortex sheet with the ends at points 1 and 2 on different sides of the cut on the orifice edge, by moving from point 2 to point 1 and taking into account the initial value of the circulation:

$$\Gamma_C = \gamma + \oint\limits_C \boldsymbol{u} \cdot d\boldsymbol{l} = \gamma + \oint\limits_C \frac{\partial \varphi}{\partial l}\, dl = \gamma + \varphi_1 - \varphi_2.$$

The difference of the potentials $\varphi_1 - \varphi_2$ can be found by the Cauchy integral [16] of the equations of motion of an ideal fluid

$$\frac{\partial \varphi_1}{\partial t} + \frac{V_1^2}{2} + \frac{p_1}{\rho} = \frac{\partial \varphi_2}{\partial t} + \frac{V_2^2}{2} + \frac{p_2}{\rho},$$

where $V_1 = 0$ and $V_2 = V_0$. As the jet boundary is a straight line near the tube edge, we have $p_1 = p_2$ (otherwise, the vortex sheet at the tube exit would deflect in the radial direction). Thus, we obtain

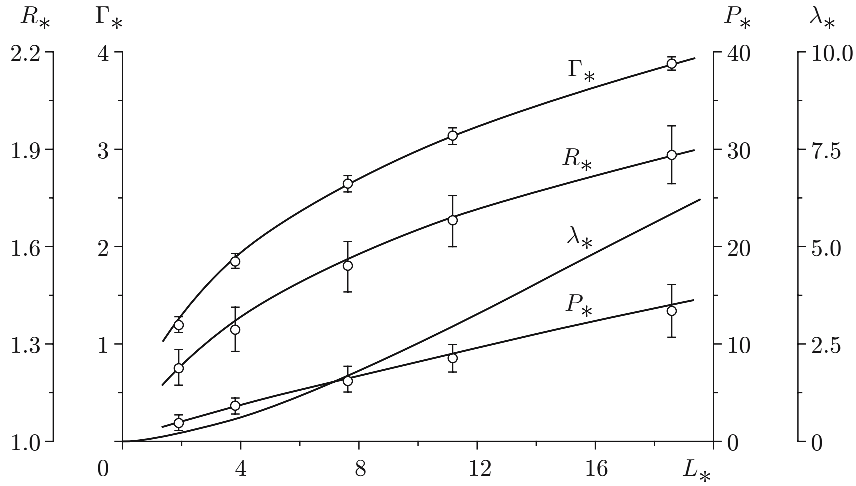

Fig. 4. Parameters R_*, Γ_*, P_*, and λ_* versus the jet length L_*: the curves and points are the calculated data and the experimental results [4], respectively.

$$\varphi_1 - \varphi_2 = \int\limits_0^T \Big(\frac{\partial \varphi_1}{\partial t} - \frac{\partial \varphi_2}{\partial t}\Big)\,dt = \frac{1}{2}\int\limits_0^T (V_2^2 - V_1^2)\,dt = \frac{V_0^2 T}{2} = \frac{V_0 L}{2}.$$

Then, the velocity circulation in the fluid (at the time $t = T$) is $\Gamma_C = \gamma + \varphi_1 - \varphi_2 = \gamma + V_0 L/2$. According to Fig. 3, the total velocity circulation in the fluid Γ_C at $t = T$ is a sum of the circulation Γ of the vortex ring and the circulation Γ_λ of the rear part of the vortex sheet of length λ, i.e., $\Gamma_C = \Gamma + \Gamma_\lambda$. We can demonstrate that $\Gamma_\lambda = \lambda V_0$. As the rear part of the vortex sheet at $t > T$ remains outside the vortex ring and dissipates, the vortex ring circulation is determined by the expression

$$\Gamma = \Gamma_C - \Gamma_\lambda = \gamma + V_0(L/2 - \lambda). \tag{4}$$

Geometric Relation. To determine R and Γ as functions of L, we need one more equation. The condition of energy conservation cannot be applied in this case, because it is difficult to estimate which part of the jet energy passes to the vortex ring and which part of the jet energy dissipates. An approximate geometric relation that follows from Fig. 3 can be used as such an equation. It was found in experiments that the velocity of propagation of the fore front of the jet is $V_0/2$; hence, the distance between the fore front of the jet and the rear part of the jet outside the vortex ring is $L/2 - \lambda$; in addition, this segment and the transverse size $d \sim 2(R - R_0)$ of the spiral structure inside the vortex ring are commensurable in magnitude (see Fig. 3). Hence, we obtain

$$L/2 - \lambda \approx 2k(R - R_0) \tag{5}$$

(k is the dimensionless proportionality coefficient of the order of unity).

Radius, Circulation, and Impulse of the Vortex Ring. In the dimensionless variables $R_* = R/R_0$, $\Gamma_* = \Gamma/(R_0 V_0)$, $\gamma_* = \gamma/(R_0 V_0)$, $L_* = L/R_0$, and $\lambda_* = \lambda/R_0$, system (3)–(5) acquires the form

$$R_*^2 \Gamma_* = L_* - \alpha\lambda_*, \quad \Gamma_* = \gamma_* + L_*/2 - \lambda_*, \quad L_*/2 - \lambda_* = 2k(R_* - 1).$$

This system yields a cubic equation for determining the radius R_* as a function of the jet length L_*:

$$R_*^3 - \Big(1 - \frac{\gamma_*}{2k}\Big)R_*^2 - \alpha R_* - \frac{1 - \alpha/2}{2k}\,L_* + \alpha = 0.$$

By solving this equation, we can determine the circulation $\Gamma_* = \gamma_* + 2k(R_* - 1)$, the impulse $P_* = R_*^2 \Gamma_*$, and the parameter $\lambda_* = L_*/2 - 2k(R_* - 1)$. The best agreement between the solutions of the above-given equations and the experimental data [4] is observed for the values of the constants $\alpha = 0.85$, $\gamma_* = 0.4$, and $k \approx 2$. The dependences of R_*, Γ_*, λ_*, and P_* on L_* obtained by the numerical solution of these equations are plotted as solid curves in Fig. 4. The points show the experimental values of R_*, Γ_*, and P_* [4]; the measurement error is also indicated. It should be noted that Eqs. (3)–(5) were derived under the assumption that $L_* \gg 1$ and cannot be used for $L_* \to 0$. As it follows from Fig. 4, however, the calculations of the vortex ring parameters in the range $1.87 \leqslant L_* \leqslant 20.00$ are in good agreement with the experimental data. For approximate estimates, the dependences of R_*, Γ_*, and P_* on the jet length L_* can be presented explicitly by approximating the dependence $\lambda_*(L_*)$ in the experimentally studied range $2 \leqslant L_* \leqslant 20$ by a certain appropriate function. For instance, the dependence $\lambda_*(L_*)$ obtained numerically is adequately approximated by a function of the form $\lambda_* = a \ln \cosh (cL_*)$ with coefficients $a \approx 2.424$ and $c \approx 0.1695$. With such an approximation by λ_*, the vortex ring parameters R_*, Γ_*, and P_* as functions of the jet length L_* can be calculated by simple formulas

$$R_* = 1 + 0.125L_* - 0.25\lambda_*, \quad \Gamma_* = 0.4 + 0.5L_* - \lambda_*, \quad P_* = L_* - 0.85\lambda_*.$$

Thus, for $\mathrm{Re} \geqslant 10^3$, the radius, circulation, and impulse of the vortex ring are determined only by the dimensionless duration of jet exhaustion L_* (or jet length) and do not depend on the jet Reynolds number Re, which is supported by the experimental data [4].

Radius of the Core of the Vortex Ring. Within the model proposed, we can derive formulas for determining one more important structural parameter of the vortex ring, vortex core radius a, as a function of L_* and Re. Obviously, comprehensive calculation of vortex core formation requires solving a rather complicated problem of rolling up of the mixing layer of the jet into the vortex core with allowance for viscosity of the medium. Nevertheless, the character of the dependence of the vortex core radius $a_* = a/R_0$ on L_* and Re can be determined without studying the process of vortex layer roll-up in detail but estimating only the volume of the fluid entering the vortex core during vortex core formation. As is seen from Fig. 3, the fluid enters the vortex core during its formation by two channels: along the mixing layer of the jet (hatched region), which is a continuation of the boundary layer flowing off from the inner surface of the nozzle, and along the branch of the spiral from the ambient space and the vortex atmosphere (non-hatched region).

Let us first estimate the volume q_1 of the fluid entering the vortex core from the mixing layer of the jet. Obviously, the velocity of the fluid inflow from the mixing layer into the vortex V_λ equals the difference between the mean velocity $\tilde{V}$ in the mixing layer and the velocity $u_\lambda = d\lambda/dt$ of axial motion of the point $z = \lambda$: $V_\lambda \approx \tilde{V} - u_\lambda$. The mean velocity averaged over the mixing layer thickness can be approximately taken as $\tilde{V} \approx V_0/2$; hence, the velocity of fluid inflow from the mixing layer of the jet into the vortex core is $V_\lambda \approx \tilde{V} - u_\lambda = V_0/2 - d\lambda/dt$. In studying the exhaustion of submerged jets, the flow in the mixing layer of the jet at moderate distances from the nozzle exit (within several nozzle radii) was found to be laminar [17, 18]. Therefore, the mixing layer thickness δ_1 at the vortex entrance can be estimated by the relation $\delta_1 \sim b\sqrt{\nu t_1}$ [18], where b is the dimensionless proportionality coefficient and $t_1 \sim (l + \lambda)/V_0$ is the time of displacement of the fluid particles to a distance $l + \lambda$. Thus, we obtain $\delta_1 \sim b\sqrt{\nu(l+\lambda)/V_0} = b\sqrt{R_0(l+\lambda)}\ \mathrm{Re}^{-1/2}$. The coefficient b can be found from the velocity distribution in the mixing layer of the jet [17, 18]. By approximating the velocity distribution in the mixing layer by a linear function, we can find the value $b \approx 5.22$. Hence, the volume q_1 of the fluid inflowing during the time T from the mixing layer into the vortex core is described by the expression

$$q_1 \approx \int_0^T 2\pi R_0 \delta_1 V_\lambda \, dt = \pi b R_0 \,\mathrm{Re}^{-1/2} \int_0^T \sqrt{R_0(l+\lambda)} \Big(V_0 - 2\,\frac{d\lambda}{dt} \Big)\, dt$$

$$= \pi b R_0 \,\mathrm{Re}^{-1/2} \int_0^L \sqrt{R_0(l+\lambda)} \Big(1 - 2\,\frac{d\lambda}{dz} \Big)\, dz,$$

where $z = V_0 t$ and $L = V_0 T$.

As was noted above, in addition to the volume q_1, a flow of a non-swirled fluid also enters the vortex core (non-hatched branch of the spiral in Fig. 3). It is difficult to determine the volume q_2 of this flow without calculating the process of rolling up of the mixing layer of the jet into a toroidal spiral surface. Based on Fig. 3, however, we can assume that the thickness of the both branches of the spiral (hatched and non-hatched regions) is approximately identical; moreover, the velocities of fluid inflow along both branches of the spiral are also identical. In the first approximation, therefore, the volume q_2 of the fluid entering the vortex core (non-hatched branch of the spiral in Fig. 3) is proportional to q_1, i.e., $q_2 \sim \beta q_1$ (β is an empirical coefficient). Thus, the total volume of the fluid entering

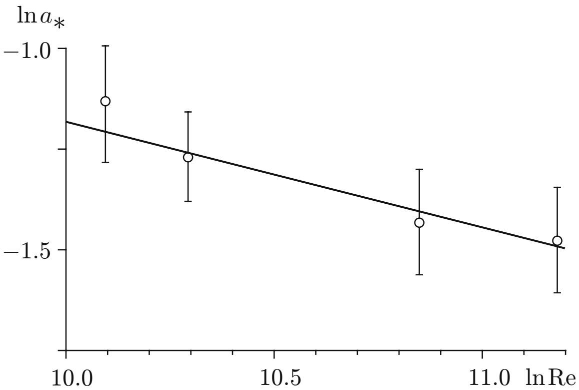

Fig. 5

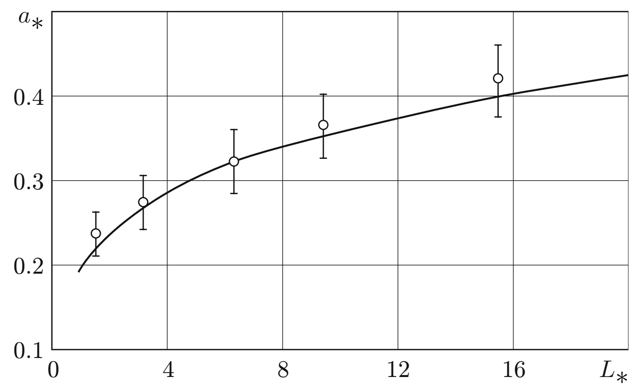

Fig. 6

Fig. 5. Vortex core radius versus the Reynolds number of the jet: the curve and points are the calculated results and the experimental data [4], respectively.

Fig. 6. Vortex core radius versus the jet length: the curve and the points are the calculated results and the experimental data [4], respectively.

the vortex core is determined by the formula $q \approx (1+\beta)q_1$. By equating the volume q of the fluid entering the vortex core to the volume of the toroidal core of the vortex ring $q_{\rm core} = \pi a^2\, 2\pi R$, we obtain

$$a \sim \sqrt{\frac{(1+\beta)b}{2\pi R}}\,\mathrm{Re}^{-1/4}\Big[R_0\int\limits_0^L \sqrt{R_0(l+\lambda)}\,\Big(1-2\,\frac{d\lambda}{dz}\Big)\,dz\Big]^{1/2}.$$

Using the dimensionless parameters $a_* = a/R_0$, $R_* = R/R_0$, $l_* = l/R_0$, $\lambda_* = \lambda/R_0$, $L_* = L/R_0$, and $z_* = z/R_0$, we can write the formula for determining the vortex radius a_* in the form

$$a_* = \frac{a}{R_0} = \mathrm{Re}^{-1/4}\sqrt{\frac{(1+\beta)b}{2\pi R_*}}\,\Big(\int\limits_0^{L_*}\sqrt{l_*+\lambda_*}\,dz_* - \frac{4}{3}\,[(l_*+\lambda_*)^{3/2} - l_*^{3/2}]\Big)^{1/2}. \qquad (6)$$

In Eq. (6), the functions $R_*(L_*)$ and $\lambda_*(L_*)$ for $L_* \geqslant 1.87$ were determined above. In integrating Eq. (6), we assumed that $\lambda_*(0) = 0$, because the length of the rear part of the jet equals zero for a zero length of the jet, as is predicted by Fig. 3. It follows from Fig. 4 that $\lambda_*(2) < 0.14$, i.e., the values of the parameter λ_* for $0 \leqslant L_* < 2$ are smaller than those for $L_* \geqslant 2$. Therefore, we can assume that the form of the function $\lambda_*(L_*)$ for $L_* < 2$ does not exert any significant influence on the values of a_* calculated for $L_* \geqslant 2$ if the values of λ_* are sufficiently high. Clearly, the vortex core radii a_* calculated by Eq. (6) are only applicable in the range $1.87 \leqslant L_* \leqslant 18.53$ where Eqs. (3)–(5) are valid (see Fig. 4). In this range, the best matching between the values of a_* calculated by Eq. (6) and the experimental data [4] is reached for $\beta \approx 3.234$. This means that the volume of the fluid entering the vortex core along the branch of the spiral that is not hatched in Fig. 3 is approximately three times the volume of the fluid entering from the mixing layer of the jet.

The power-law dependence of a_* on the Reynolds number Re is in good agreement with the dependence of $\ln a_*$ on $\ln \mathrm{Re}$ plotted on the basis of the experimental data [4] (Fig. 5). The tangent of the angle of the straight line plotted through the experimental points in Fig. 5 is $-1/4$, which corresponds to Eq. (6). It was also noted [6, 7] that the measured results predicted the dependence of the form $a \sim \mathrm{Re}^{-1/4}$. The dependence of a_* on the jet length L_* [see Eq. (6)] ($\mathrm{Re} = 1.825 \cdot 10^4$ and $l_* = 2$) is plotted in Fig. 6 by the solid curve; the points are the experimental data [4]. Equation (6) also determines the character of the dependence of a_* on the length l_* of the cylindrical segment of the nozzle, though no experimental data are available for this case. In using Eq. (6) for calculations, one should take into account that the formula was derived under the assumption of a small thickness of the mixing layer of the jet δ_1, as compared with the jet radius R_0; hence, the formula is applicable for $\delta_1/R_0 \sim \sqrt{(l_*+\lambda_*)/\mathrm{Re}} \ll 1$, which corresponds to conditions of formation of turbulent vortex rings with a thin core at sufficiently high Reynolds numbers of the jet.

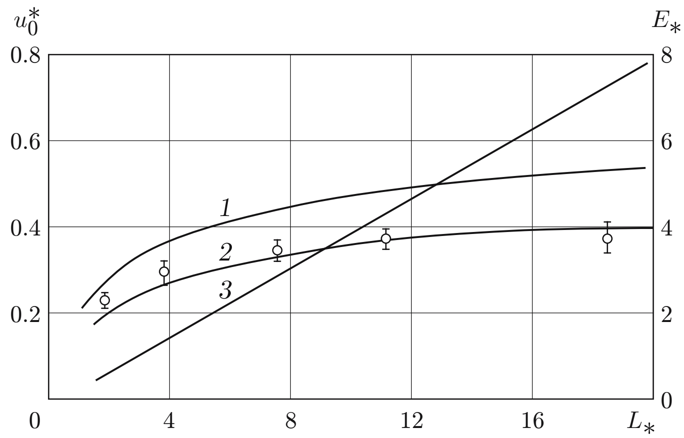

Fig. 7. Translational velocity (1 and 2) and energy (3) of the vortex ring versus the jet length.

Translational Velocity of the Vortex Ring. The results obtained above allow us to calculate the translational velocity of the vortex ring u_0 by known theoretical formulas [1, 2], for instance, by the Kelvin formula

$$u_0 = \frac{\Gamma}{4\pi R}\Big(\ln\frac{8R}{a} - \frac{1}{4}\Big).$$

As the vortex core radius a in this formula is a function of the jet length L_* and Reynolds number Re, the velocity of the vortex ring also depends on these parameters. It is clear that u_0 depends on Re only weakly, and it is only the dependence of u_0 on the jet length L_* that is important. Obtaining expressions for calculating all parameters in this formula, we can find the translational velocity of the vortex as a function of L_*. The calculated dependence of $u_0^* = u_0/V_0$ on L_* is plotted by curve 1 in Fig. 7; the points correspond to the experimental values of u_0^* borrowed from [4]. It is seen that the Kelvin formula overpredicts the values of u_0^*, even though the values of all parameters used in this formula (R, Γ, and a) are consistent with the experiment. This fact suggests that the formulas for the translational velocity of the vortex ring, derived within the framework of an ideal fluid under the assumption that the vorticity is focused in the vortex core only, cannot provide accurate quantitative estimates for the velocity of real vortex rings with a smooth bell-shaped distribution of vorticity extended outside the vortex core as well. Speaking globally, the notion of the vortex core is not defined clearly enough for real vortex cores with smooth distributions of vorticity; this notion is used only to characterize the width of the peak in the vorticity distribution and to describe the vortex ring structure by a finite number of parameters. Obviously, a more profound theory is needed to calculate the translational velocity of real vortex rings. It should be noted, however, that the agreement between the values of u_0^* calculated by the Kelvin formula and the experimental data can be improved by using a correction $\alpha \approx 0.75$ in this formula. The dependence of u_0^* on the jet length L_* calculated with the use of this correction is plotted by curve 2 in Fig. 7.

Vortex Ring Energy. Available theoretical models of the vortex ring [1–3] predict that the energy of the vortex ring with a thin core is determined as a function of the parameters R, Γ, and a:

$$E = \frac{\rho R\Gamma^2}{2}\Big(\ln\frac{8R}{a} - \frac{7}{4}\Big).$$

Obviously, the dimensionless value of the energy $E_* = E/(\rho\pi R_0^3 V_0^2) = (1/2\pi)R_*\Gamma_*^2[\ln(8R/a) - 7/4]$, like the vortex ring velocity u_0^*, depends only weakly on the Reynolds number of the jet. An analysis of the calculated dependence of E_* on the jet length L_*, plotted by curve 3 in Fig. 7, shows that E_* changes almost proportional to L_* ($E_* \sim 0.4L_*$). We can assume that the calculated dependence of E_* describes the qualitative behavior of the vortex ring energy as a function of the jet length. It is difficult, however, to check the quantitative correctness of the calculation, because no experimental data on the dependence of E_* on the jet length L_* are available.

Thus, the proposed model of vortex ring formation, which was constructed with the use of only one empirical constant for each functional dependence to be determined, allows one to calculate almost all characteristics of the vortex ring formed in the case of pulsed exhaustion of a submerged jet and to predict the vortex ring structure.

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